

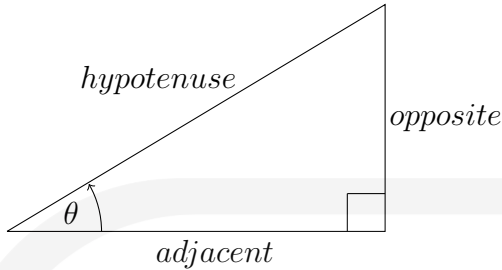
# Trigonometric Formula Sheet

## Definition of the Trig Functions

### Right Triangle Definition

Assume that:

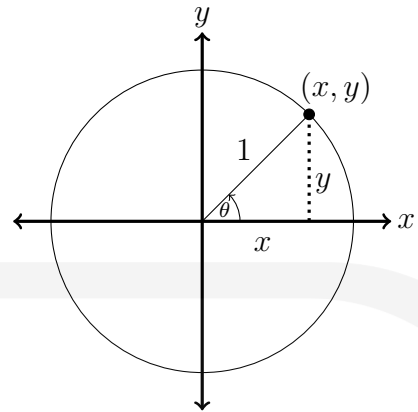
$$0 < \theta < \frac{\pi}{2} \quad \text{or} \quad 0^\circ < \theta < 90^\circ$$



$$\begin{aligned} \sin \theta &= \frac{\text{opp}}{\text{hyp}} & \csc \theta &= \frac{\text{hyp}}{\text{opp}} \\ \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} & \cot \theta &= \frac{\text{adj}}{\text{opp}} \end{aligned}$$

### Unit Circle Definition

Assume  $\theta$  can be any angle.



$$\begin{aligned} \sin \theta &= \frac{y}{1} & \csc \theta &= \frac{1}{y} \\ \cos \theta &= \frac{x}{1} & \sec \theta &= \frac{1}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

### Domains of the Trig Functions

$$\sin \theta, \quad \forall \theta \in (-\infty, \infty)$$

$$\csc \theta, \quad \forall \theta \neq n\pi, \text{ where } n \in \mathbb{Z}$$

$$\cos \theta, \quad \forall \theta \in (-\infty, \infty)$$

$$\sec \theta, \quad \forall \theta \neq \left(n + \frac{1}{2}\right)\pi, \text{ where } n \in \mathbb{Z}$$

$$\tan \theta, \quad \forall \theta \neq \left(n + \frac{1}{2}\right)\pi, \text{ where } n \in \mathbb{Z}$$

$$\cot \theta, \quad \forall \theta \neq n\pi, \text{ where } n \in \mathbb{Z}$$

### Ranges of the Trig Functions

$$-1 \leq \sin \theta \leq 1$$

$$\csc \theta \geq 1 \text{ and } \csc \theta \leq -1$$

$$-1 \leq \cos \theta \leq 1$$

$$\sec \theta \geq 1 \text{ and } \sec \theta \leq -1$$

$$-\infty \leq \tan \theta \leq \infty$$

$$-\infty \leq \cot \theta \leq \infty$$

### Periods of the Trig Functions

The period of a function is the number,  $T$ , such that  $f(\theta + T) = f(\theta)$ .

So, if  $\omega$  is a fixed number and  $\theta$  is any angle we have the following periods.

$$\sin(\omega\theta) \Rightarrow T = \frac{2\pi}{\omega}$$

$$\csc(\omega\theta) \Rightarrow T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \Rightarrow T = \frac{2\pi}{\omega}$$

$$\sec(\omega\theta) \Rightarrow T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \Rightarrow T = \frac{\pi}{\omega}$$

$$\cot(\omega\theta) \Rightarrow T = \frac{\pi}{\omega}$$

# Identities and Formulas

## Tangent and Cotangent Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

## Reciprocal Identities

$$\begin{aligned} \sin \theta &= \frac{1}{\csc \theta} & \csc \theta &= \frac{1}{\sin \theta} \\ \cos \theta &= \frac{1}{\sec \theta} & \sec \theta &= \frac{1}{\cos \theta} \\ \tan \theta &= \frac{1}{\cot \theta} & \cot \theta &= \frac{1}{\tan \theta} \end{aligned}$$

## Pythagorean Identities

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta \end{aligned}$$

## Even and Odd Formulas

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \csc(-\theta) &= -\csc \theta \\ \cos(-\theta) &= \cos \theta & \sec(-\theta) &= \sec \theta \\ \tan(-\theta) &= -\tan \theta & \cot(-\theta) &= -\cot \theta \end{aligned}$$

## Periodic Formulas

If  $n$  is an integer

$$\begin{aligned} \sin(\theta + 2\pi n) &= \sin \theta & \csc(\theta + 2\pi n) &= \csc \theta \\ \cos(\theta + 2\pi n) &= \cos \theta & \sec(\theta + 2\pi n) &= \sec \theta \\ \tan(\theta + \pi n) &= \tan \theta & \cot(\theta + \pi n) &= \cot \theta \end{aligned}$$

## Double Angle Formulas

$$\begin{aligned} \sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta \\ \tan(2\theta) &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

## Degrees to Radians Formulas

If  $x$  is an angle in degrees and  $t$  is an angle in radians then:

$$\frac{\pi}{180^\circ} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180^\circ} \quad \text{and} \quad x = \frac{180^\circ t}{\pi}$$

## Half Angle Formulas

$$\begin{aligned} \sin \theta &= \pm \sqrt{\frac{1 - \cos(2\theta)}{2}} \\ \cos \theta &= \pm \sqrt{\frac{1 + \cos(2\theta)}{2}} \\ \tan \theta &= \pm \sqrt{\frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}} \end{aligned}$$

## Sum and Difference Formulas

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \\ \cos(\alpha \pm \beta) &= \cos \alpha \cos \beta \mp \sin \alpha \sin \beta \\ \tan(\alpha \pm \beta) &= \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta} \end{aligned}$$

## Product to Sum Formulas

$$\begin{aligned} \sin \alpha \sin \beta &= \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \\ \cos \alpha \cos \beta &= \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] \\ \sin \alpha \cos \beta &= \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] \\ \cos \alpha \sin \beta &= \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)] \end{aligned}$$

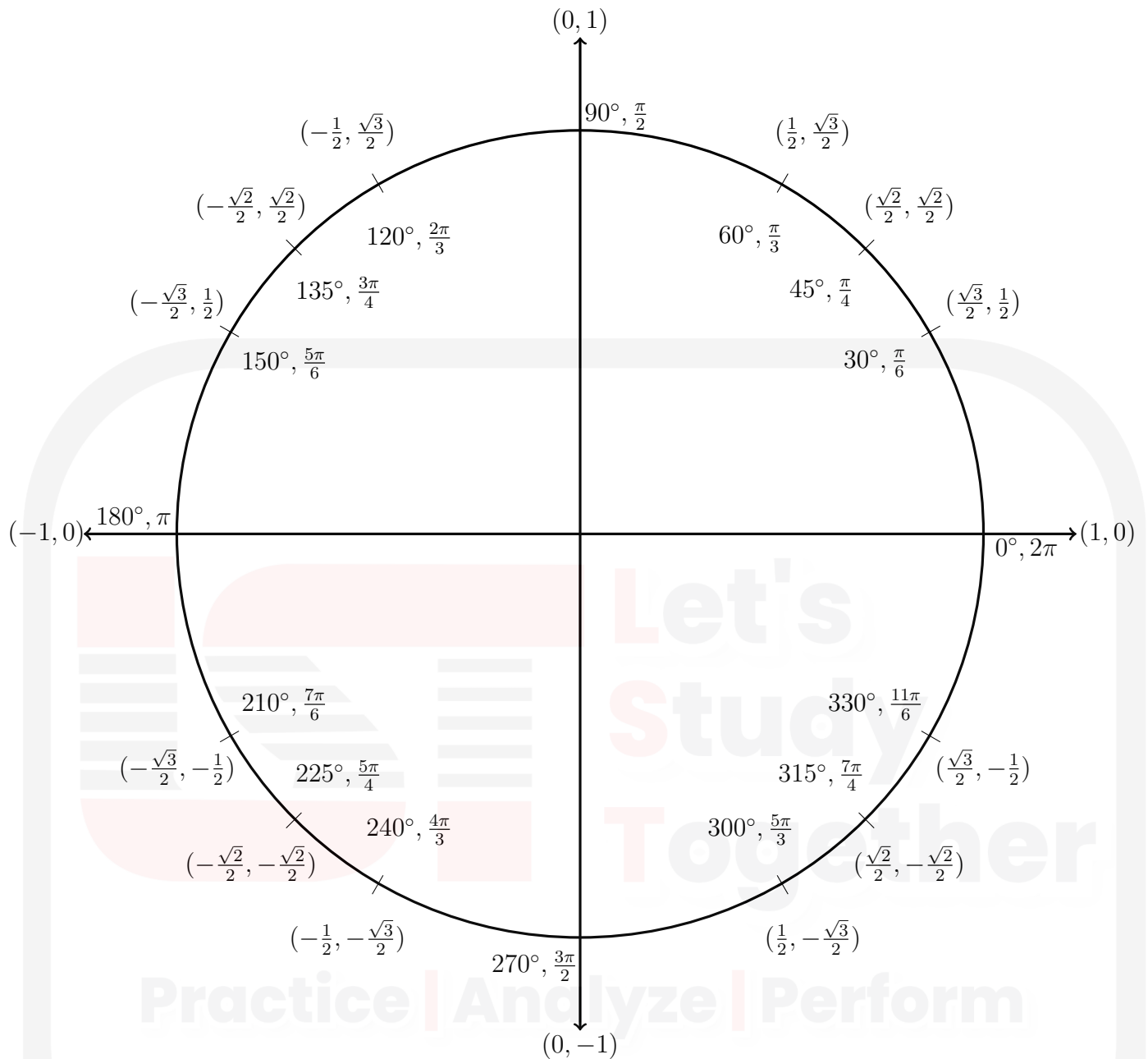
## Sum to Product Formulas

$$\begin{aligned} \sin \alpha + \sin \beta &= 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) \\ \sin \alpha - \sin \beta &= 2 \cos \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right) \\ \cos \alpha + \cos \beta &= 2 \cos \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) \\ \cos \alpha - \cos \beta &= -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right) \end{aligned}$$

## Cofunction Formulas

$$\begin{aligned} \sin \left( \frac{\pi}{2} - \theta \right) &= \cos \theta & \cos \left( \frac{\pi}{2} - \theta \right) &= \sin \theta \\ \csc \left( \frac{\pi}{2} - \theta \right) &= \sec \theta & \sec \left( \frac{\pi}{2} - \theta \right) &= \csc \theta \\ \tan \left( \frac{\pi}{2} - \theta \right) &= \cot \theta & \cot \left( \frac{\pi}{2} - \theta \right) &= \tan \theta \end{aligned}$$

# Unit Circle



For any ordered pair on the unit circle  $(x, y)$  :  $\cos \theta = x$  and  $\sin \theta = y$

## Example

$$\cos \left( \frac{7\pi}{6} \right) = -\frac{\sqrt{3}}{2} \quad \sin \left( \frac{7\pi}{6} \right) = -\frac{1}{2}$$

# Inverse Trig Functions

## Definition

$\theta = \sin^{-1}(x)$  is equivalent to  $x = \sin \theta$

$\theta = \cos^{-1}(x)$  is equivalent to  $x = \cos \theta$

$\theta = \tan^{-1}(x)$  is equivalent to  $x = \tan \theta$

## Inverse Properties

These properties hold for  $x$  in the domain and  $\theta$  in the range

$$\sin(\sin^{-1}(x)) = x$$

$$\sin^{-1}(\sin(\theta)) = \theta$$

$$\cos(\cos^{-1}(x)) = x$$

$$\cos^{-1}(\cos(\theta)) = \theta$$

$$\tan(\tan^{-1}(x)) = x$$

$$\tan^{-1}(\tan(\theta)) = \theta$$

## Domain and Range

### Function

### Domain

### Range

### Other Notations

$$\theta = \sin^{-1}(x) \quad -1 \leq x \leq 1 \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\sin^{-1}(x) = \arcsin(x)$$

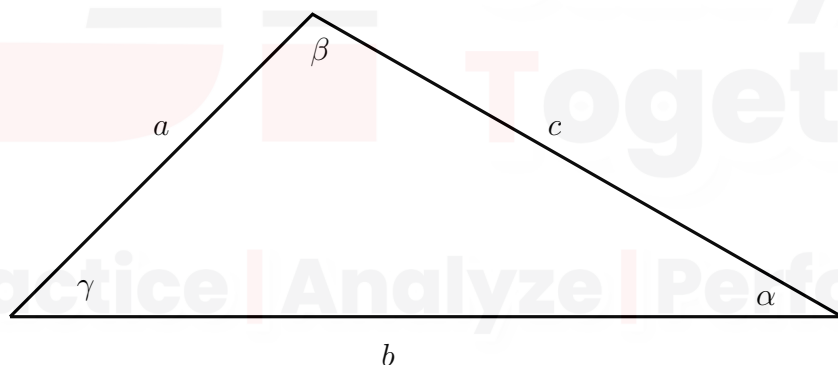
$$\theta = \cos^{-1}(x) \quad -1 \leq x \leq 1 \quad 0 \leq \theta \leq \pi$$

$$\cos^{-1}(x) = \arccos(x)$$

$$\theta = \tan^{-1}(x) \quad -\infty \leq x \leq \infty \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\tan^{-1}(x) = \arctan(x)$$

## Law of Sines, Cosines, and Tangents



### Law of Sines

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

### Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

### Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}$$

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(\beta - \gamma)}{\tan \frac{1}{2}(\beta + \gamma)}$$

$$\frac{a-c}{a+c} = \frac{\tan \frac{1}{2}(\alpha - \gamma)}{\tan \frac{1}{2}(\alpha + \gamma)}$$

## Complex Numbers

$$i = \sqrt{-1} \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

$$\sqrt{-a} = i\sqrt{a}, \quad a \geq 0$$

$$(a + bi)(a - bi) = a^2 + b^2$$

$$(a + bi) + (c + di) = a + c + (b + d)i$$

$$|a + bi| = \sqrt{a^2 + b^2} \quad \text{Complex Modulus}$$

$$(a + bi) - (c + di) = a - c + (b - d)i$$

$$\overline{(a + bi)} = a - bi \quad \text{Complex Conjugate}$$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

$$\overline{(a + bi)}(a + bi) = |a + bi|^2$$

### DeMoivre's Theorem

Let  $z = r(\cos \theta + i \sin \theta)$ , and let  $n$  be a positive integer.

Then:

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

**Example:** Let  $z = 1 - i$ , find  $z^6$ .

Solution: First write  $z$  in polar form.

$$r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \arg(z) = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

$$\text{Polar Form: } z = \sqrt{2} \left( \cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right)$$

Applying DeMoivre's Theorem gives :

$$z^6 = (\sqrt{2})^6 \left( \cos\left(6 \cdot -\frac{\pi}{4}\right) + i \sin\left(6 \cdot -\frac{\pi}{4}\right) \right)$$

$$= 2^3 \left( \cos\left(-\frac{3\pi}{2}\right) + i \sin\left(-\frac{3\pi}{2}\right) \right)$$

$$= 8(0 + i(1))$$

$$= 8i$$

## Finding the $n$ th roots of a number using DeMoivre's Theorem

**Example:** Find all the complex fourth roots of 4. That is, find all the complex solutions of  $x^4 = 4$ .

We are asked to find all complex fourth roots of 4.

These are all the solutions (including the complex values) of the equation  $x^4 = 4$ .

For any positive integer  $n$ , a nonzero complex number  $z$  has exactly  $n$  distinct  $n$ th roots. More specifically, if  $z$  is written in the trigonometric form  $r(\cos \theta + i \sin \theta)$ , the  $n$ th roots of  $z$  are given by the following formula.

$$(*) \quad r^{\frac{1}{n}} \left( \cos \left( \frac{\theta}{n} + \frac{360^\circ k}{n} \right) + i \sin \left( \frac{\theta}{n} + \frac{360^\circ k}{n} \right) \right), \quad \text{for } k = 0, 1, 2, \dots, n - 1.$$

Remember from the previous example we need to write 4 in trigonometric form by using:

$$r = \sqrt{(a)^2 + (b)^2} \quad \text{and} \quad \theta = \arg(z) = \tan^{-1} \left( \frac{b}{a} \right).$$

So we have the complex number  $a + ib = 4 + i0$ .

Therefore  $a = 4$  and  $b = 0$

So  $r = \sqrt{(4)^2 + (0)^2} = 4$  and

$$\theta = \arg(z) = \tan^{-1} \left( \frac{0}{4} \right) = 0$$

Finally our trigonometric form is  $4 = 4(\cos 0^\circ + i \sin 0^\circ)$

Using the formula (\*) above with  $n = 4$ , we can find the fourth roots of  $4(\cos 0^\circ + i \sin 0^\circ)$

- For  $k = 0$ ,  $4^{\frac{1}{4}} \left( \cos \left( \frac{0^\circ}{4} + \frac{360^\circ * 0}{4} \right) + i \sin \left( \frac{0^\circ}{4} + \frac{360^\circ * 0}{4} \right) \right) = \sqrt{2} (\cos(0^\circ) + i \sin(0^\circ)) = \sqrt{2}$
- For  $k = 1$ ,  $4^{\frac{1}{4}} \left( \cos \left( \frac{0^\circ}{4} + \frac{360^\circ * 1}{4} \right) + i \sin \left( \frac{0^\circ}{4} + \frac{360^\circ * 1}{4} \right) \right) = \sqrt{2} (\cos(90^\circ) + i \sin(90^\circ)) = \sqrt{2}i$
- For  $k = 2$ ,  $4^{\frac{1}{4}} \left( \cos \left( \frac{0^\circ}{4} + \frac{360^\circ * 2}{4} \right) + i \sin \left( \frac{0^\circ}{4} + \frac{360^\circ * 2}{4} \right) \right) = \sqrt{2} (\cos(180^\circ) + i \sin(180^\circ)) = -\sqrt{2}$
- For  $k = 3$ ,  $4^{\frac{1}{4}} \left( \cos \left( \frac{0^\circ}{4} + \frac{360^\circ * 3}{4} \right) + i \sin \left( \frac{0^\circ}{4} + \frac{360^\circ * 3}{4} \right) \right) = \sqrt{2} (\cos(270^\circ) + i \sin(270^\circ)) = -\sqrt{2}i$

Thus all of the complex roots of  $x^4 = 4$  are:

$$\sqrt{2}, \sqrt{2}i, -\sqrt{2}, -\sqrt{2}i .$$

# Formulas for the Conic Sections

## Circle

$$\text{Standard Form : } (x - h)^2 + (y - k)^2 = r^2$$

Where  $(h, k)$  = center and  $r$  = radius

## Ellipse

Standard Form for Horizontal Major Axis :

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Standard Form for Vertical Major Axis :

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

Where  $(h, k)$  = center

$2a$  = length of major axis

$2b$  = length of minor axis

$$(0 < b < a)$$

Foci can be found by using  $c^2 = a^2 - b^2$

Where  $c$  = foci length

# More Conic Sections

## Hyperbola

*Standard Form for Horizontal Transverse Axis :*

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

*Standard Form for Vertical Transverse Axis :*

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

Where  $(h, k)$  = center

$a$  = distance between center and either vertex

Foci can be found by using  $b^2 = c^2 - a^2$

Where  $c$  is the distance between

center and either focus. ( $b > 0$ )

## Parabola

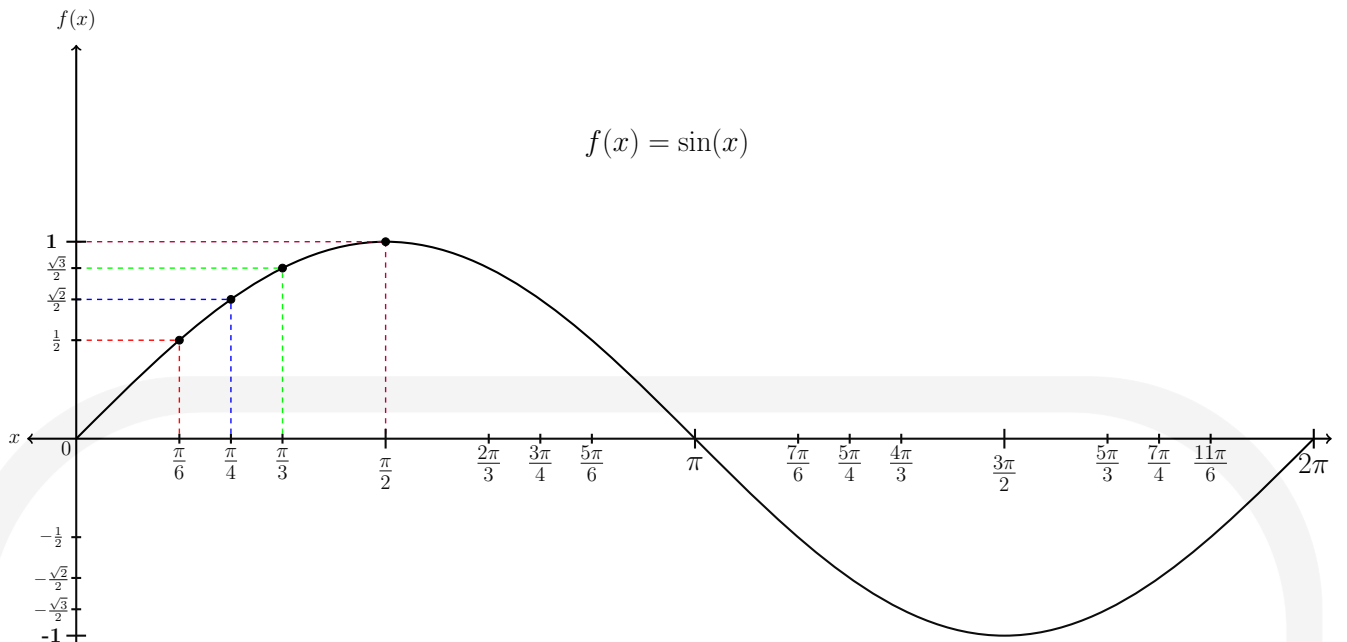
Vertical axis:  $y = a(x - h)^2 + k$

Horizontal axis:  $x = a(y - k)^2 + h$

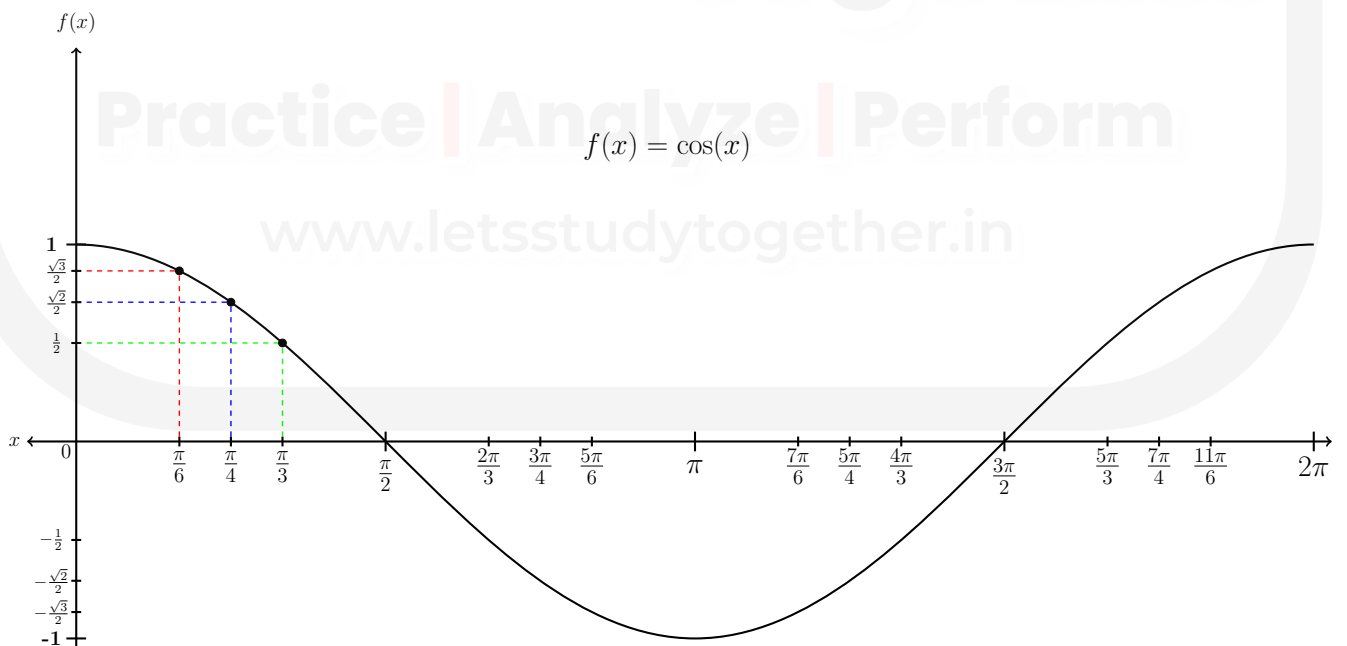
Where  $(h, k)$  = vertex

$a$  = scaling factor

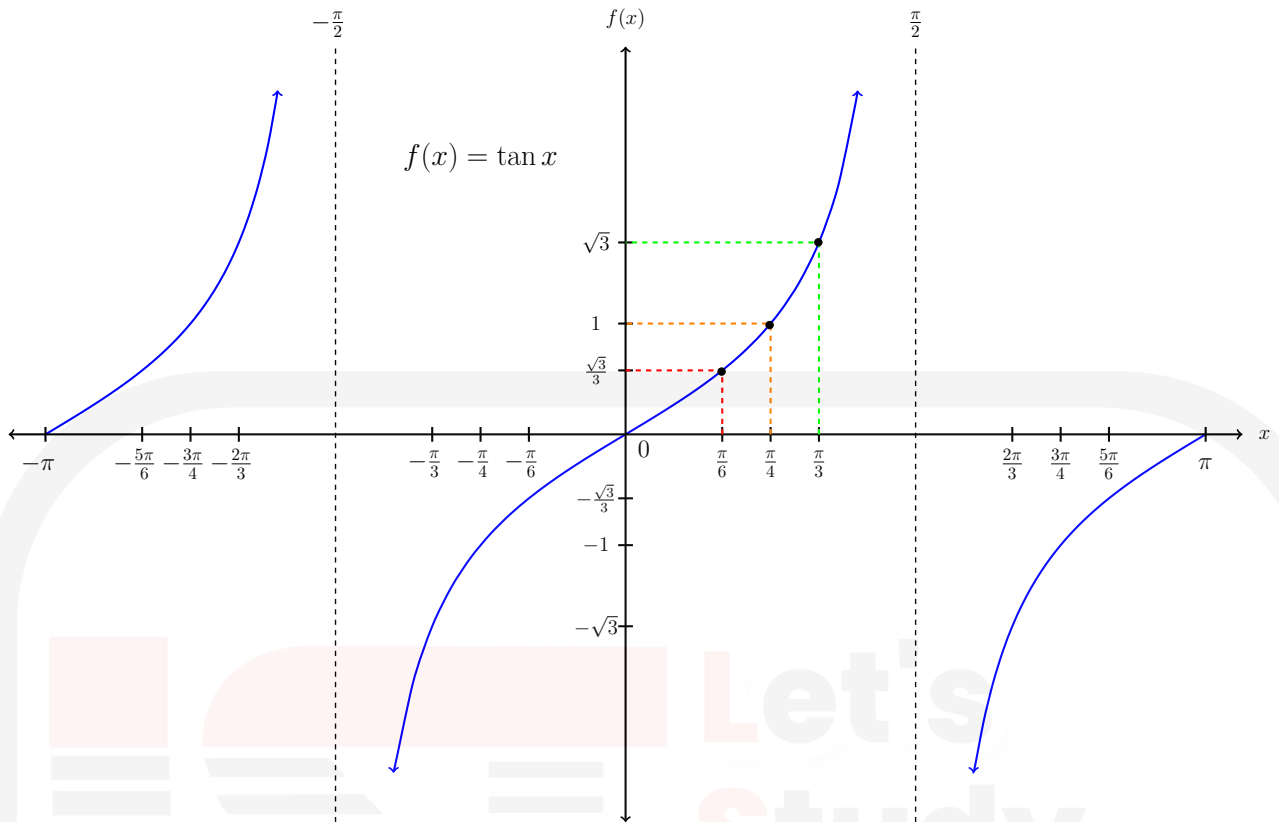




Example :  $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$



Example :  $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$



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